Plan

Recap

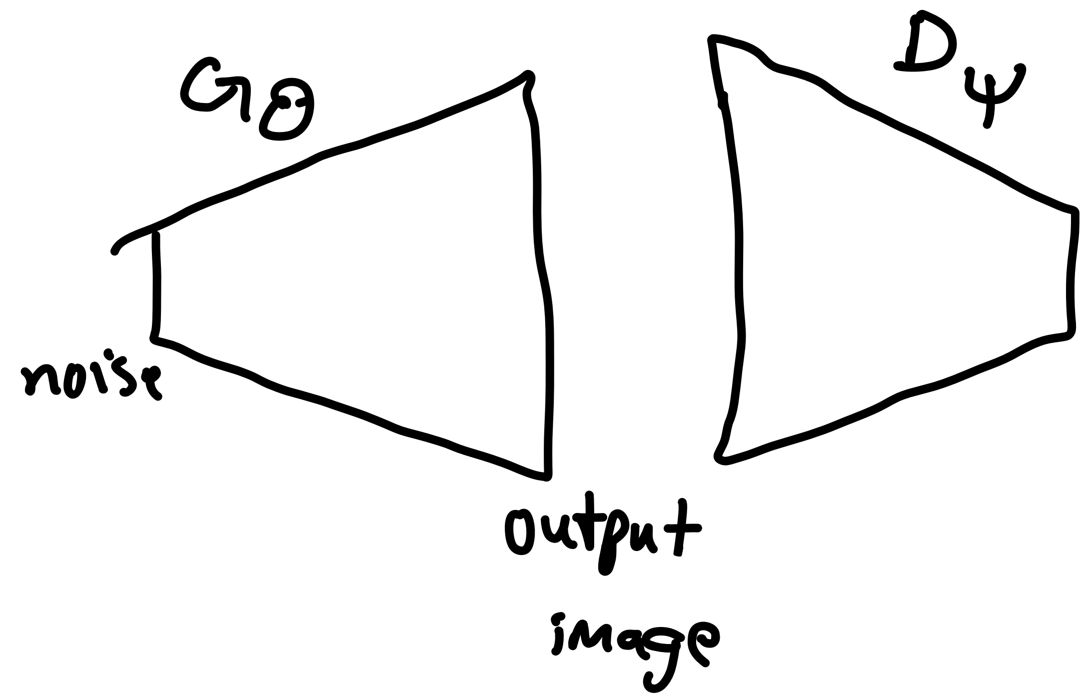
Logistics

Contrastive Learning

Math!

CLIP

Recap



mode collapse

conditional GAN

$$\mathbb{E}[x] = \sum_x x \Pr(x)$$

$$\mathcal{L}(\theta,\psi) = \mathbb{E}_{x\sim \text{real}}\left[D_\psi(x, c)\right] - \mathbb{E}_{z\sim\text{random}}\left[D_\psi(G_\theta(z, c), c)\right]$$

Logistics

Forms: 22/25

Homework due 5pm Fri

Project!

↳ minimum working example

↳ attribute code

↳ nontrivial modifications

↳ post on canvas

Contrastive Learning

Unsupervised Learning (no labels)

Lots of images without labels

↳ too many images

↳ too expensive

Goal: embed images so similar images have close embeddings

then pass to downstream

↳ label with few labels

↳ CLIP

Contrastive Learning

Positive Pair

(x, x') these are close

$$f_\theta(x) \approx f_\theta(x')$$

$$f_\theta(x)^T f_\theta(x') \text{ large}$$

Negative Pair

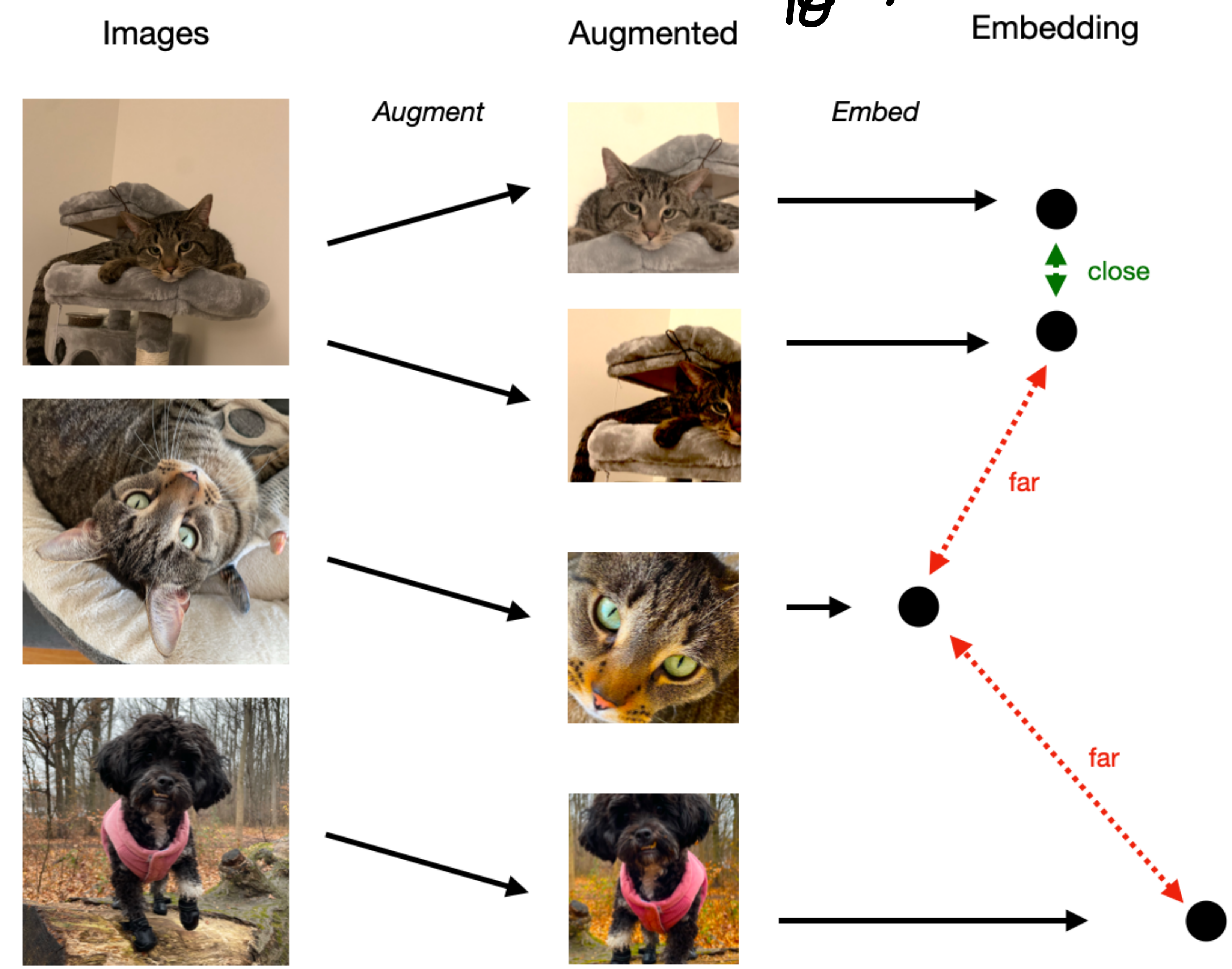
(x, x') drawn randomly

$$f_\theta(x) \neq f_\theta(x')$$

$$f_\theta(x)^T f_\theta(x') \text{ small}$$

How do we get positive pairs?

$$f_\theta : \mathbb{R}^{m \times m} \to \mathbb{R}^k$$



Loss Function

$$\mathcal{L}(\theta) = -2 \mathop{\mathbb{E}}_{x,x' \text{ positive}}\left[f_\theta(x)^T f_\theta(x') \right]$$

$$+ \mathop{\mathbb{E}}_{x,x' \text{ negative}}\left[(f_\theta(x)^T f_\theta(x'))^2 \right]$$

$$\sum_{x,x'} f_\theta(x)^T f_\theta(x)' \; p(x,x')$$

$$\sum_{x,x'} (f_\theta(x)^T f_\theta(x'))^2 \, p(x) \cdot p(x')$$

Overview

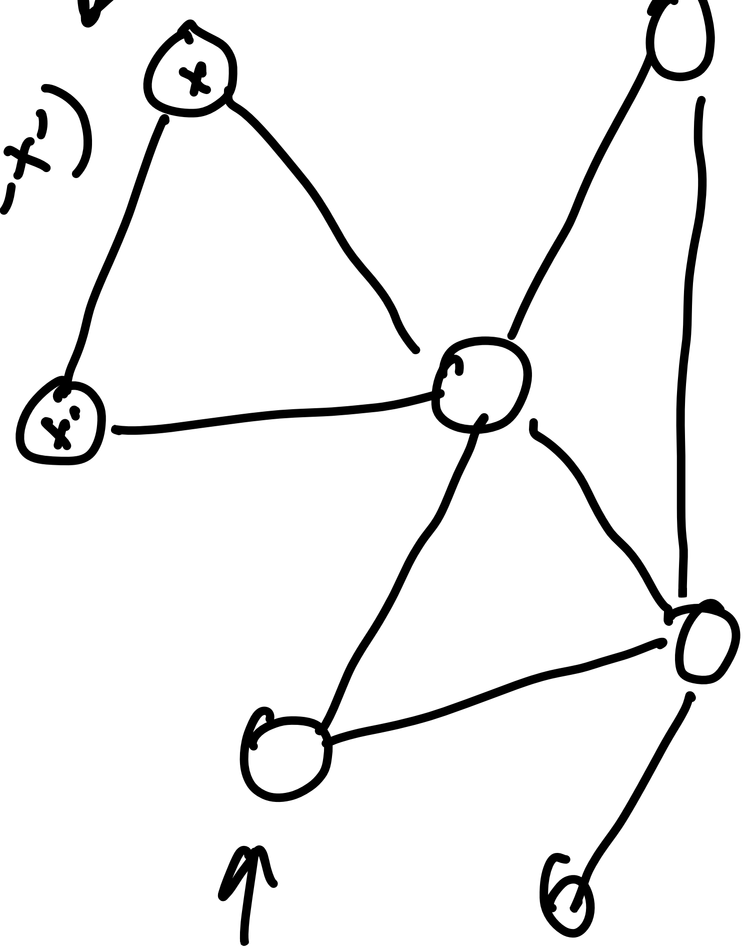
Assume training works, we will characterize optimal loss

Graph

n nodes

probability we sample x, x' as positive pair

p(x, x')



augmented image

$$
A = \begin{array}{cc} & \begin{array}{c} x' \quad\quad n \end{array} \\ \begin{array}{c} x \\ n \end{array} & \begin{bmatrix} \\ & p(x, x') & \\ & & \end{bmatrix} \end{array}
$$

$$
D = \begin{array}{cc} & \begin{array}{c} x \quad\quad n \end{array} \\ \begin{array}{c} x \\ n \end{array} & \begin{bmatrix} p(x) & & 0 \\ & \ddots & \\ 0 & & \end{bmatrix} \end{array}
$$

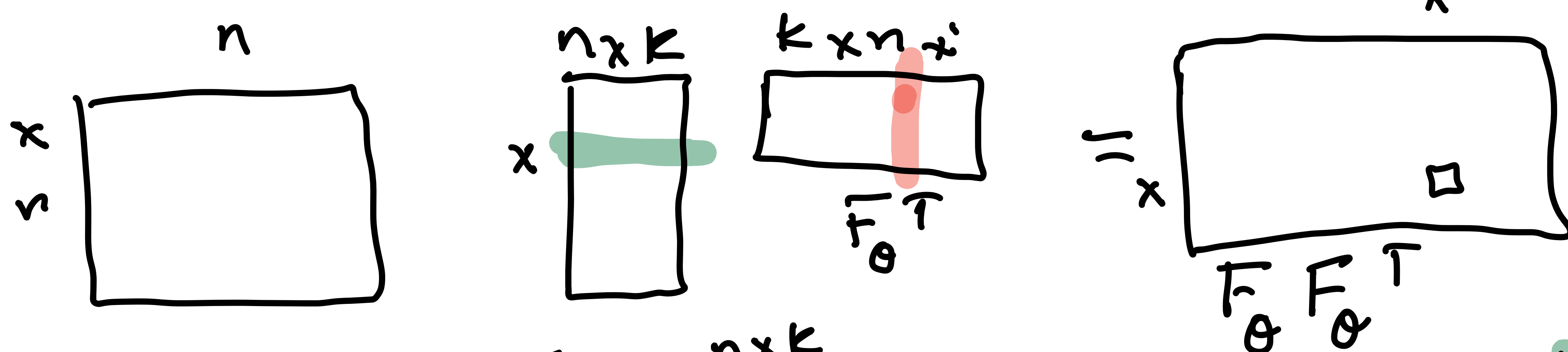
n

$$
\underset{n \times n}{\bar{A}} = \underset{n \times n}{D^{-1/2}} \, A \, \underset{n \times n}{D^{-1/2}} = n \begin{bmatrix} \\ \frac{p(x, x')}{\sqrt{p(x) \cdot p(x')}} \\ \\ \end{bmatrix}
$$

Claim

$$\operatorname*{argmin}_{\theta} \mathcal{L}(\theta) \overset{\checkmark}{=} \operatorname*{argmin}_{\theta} \|\bar{A} - F_\theta F_\theta^T\|_F^2$$

square every entry and add



$F_\theta \in \mathbb{R}^{n \times k}$

$f_\theta(x)^T \sqrt{p(x)}$

$f_\theta(x') \sqrt{p(x')}$

$$\|\bar{A} - F_\theta F_\theta^T\|_F^2 = \sum_{x,x'} \left(\frac{p(x,x')}{\sqrt{p(x)}\sqrt{p(x')}} - f_\theta(x)^T \sqrt{p(x)}\, f_\theta(x')\sqrt{p(x')} \right)^2$$

$$= \text{constant} - 2 \sum_{x,x'} p(x,x')\, f_\theta(x)^T f_\theta(x') + \sum_{x,x'} p(x)p(x') \left(f_\theta(x)^T f_\theta(x')\right)^2$$

$$= \text{constant} - 2\, \mathbb{E}_{x,x' \sim \text{positive}} \left[f_\theta(x)^T f_\theta(x') \right] + \mathbb{E}_{x,x' \sim \text{negative}} \left[\left(f_\theta(x)^T f_\theta(x')\right)^2 \right]$$

$$= \text{constant} + \mathcal{L}(\theta)$$

$$\theta^* = \underset{\theta}{\operatorname{argmin}} \; \| \bar{A} - F_\theta F_\theta^T \|_F^2$$

$$F_{\theta^*} F_{\theta^*}^T = \sum_{i=1}^{k} \lambda_i v_i v_i^T$$

orthonormal

$$\sum_{j=1}^{n} v_{i,j}^2 = 1 = \|v_i\|_2^2 = v_i^T v_i$$

$$v_i^T v_j = 0 \text{ if } i \neq j$$

$\bar{A}$ has eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq 1$$

$$v_1, v_2, v_3, \ldots, v_n$$

eigenvectors

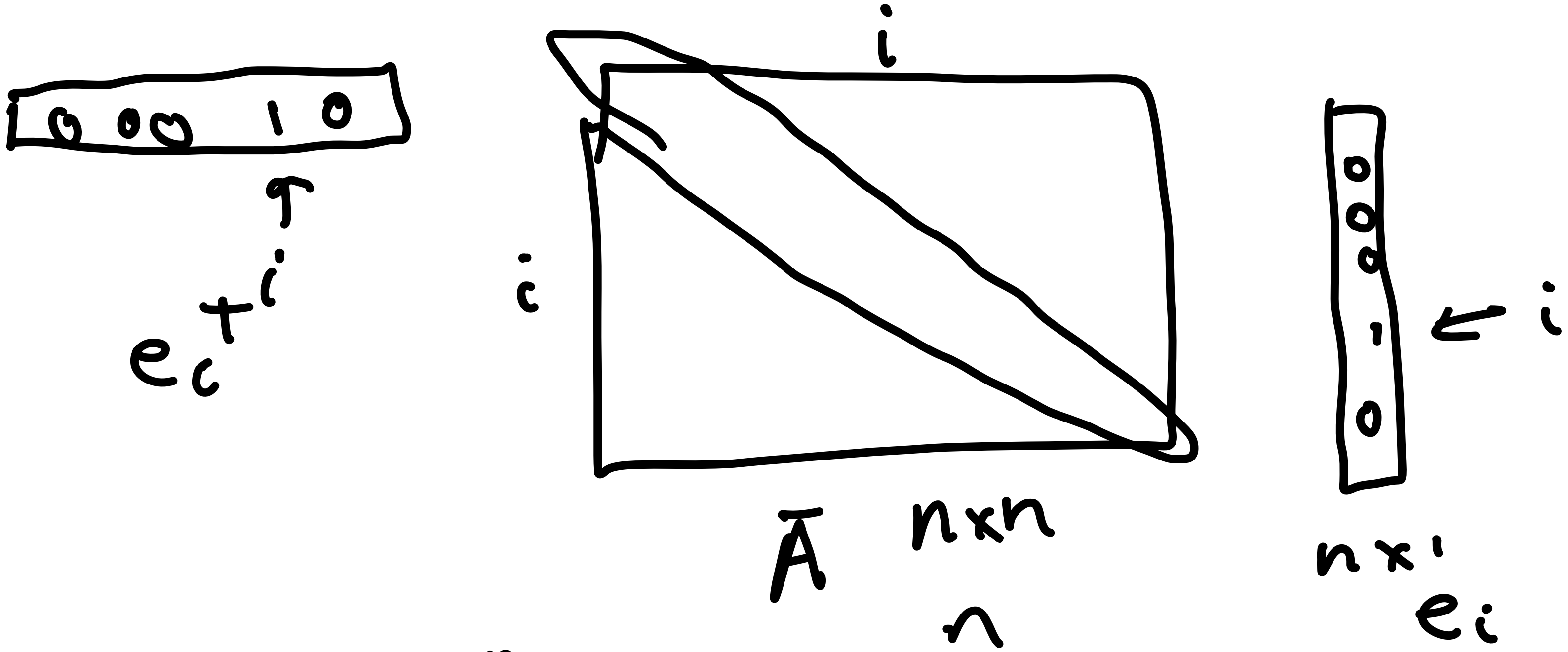
$$\bar{A} \overset{\checkmark}{=} \sum_{j=1}^{n} \underset{n \times 1}{v_j} \; \underset{1 \times n}{v_j^T} \lambda_j$$

$$\bar{A} v_i = \lambda_i v_i \qquad i \in [n]$$

$$\Rightarrow \quad \sum_{j=1}^{n} v_j v_j^T \lambda_j \; v_i = \sum_{j=1}^{n} \lambda_j v_j v_j^T v_i$$

$$= 0 + 0 + 0 + \ldots + \lambda_i v_i \cdot 1 + 0$$

$$= \lambda_i v_i$$

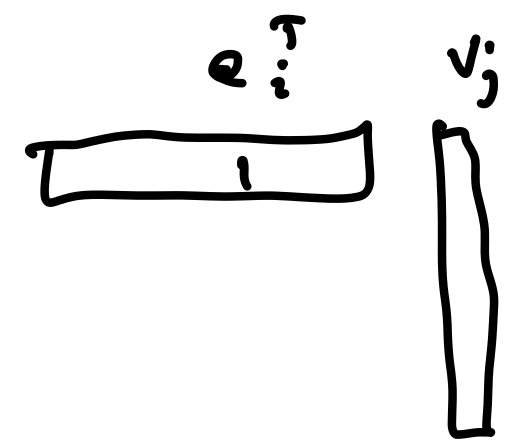


$$= [0\,0\,1\,0] \begin{bmatrix} \\ \\ \end{bmatrix} \leftarrow i \quad = \hat{A}_{i,i}$$

$\bar{A}_{,i}$

$n \times 1$

$$tr(\bar{A}) = \sum_{i=1}^{n} \bar{A}_{i,i} = \sum_{i=1}^{n} \lambda_i \leftarrow \text{want to show}$$



$$\sum_{i=1}^{n} \bar{A}_{i,i} = \sum_{i=1}^{n} e_i^T (\bar{A}) e_i = \sum_{i=1}^{n} e_i^T \Big(\sum_{j=1}^{n} v_j v_j^T \lambda_j\Big) e_i$$

$$= \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} e_i^T v_j v_j^T e_i$$

$$= \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} v_{j,i}^2 = \sum_{j=1}^{n} \lambda_j \|v_j\|_2^2 = \sum_{j=1}^{n} \lambda_j$$

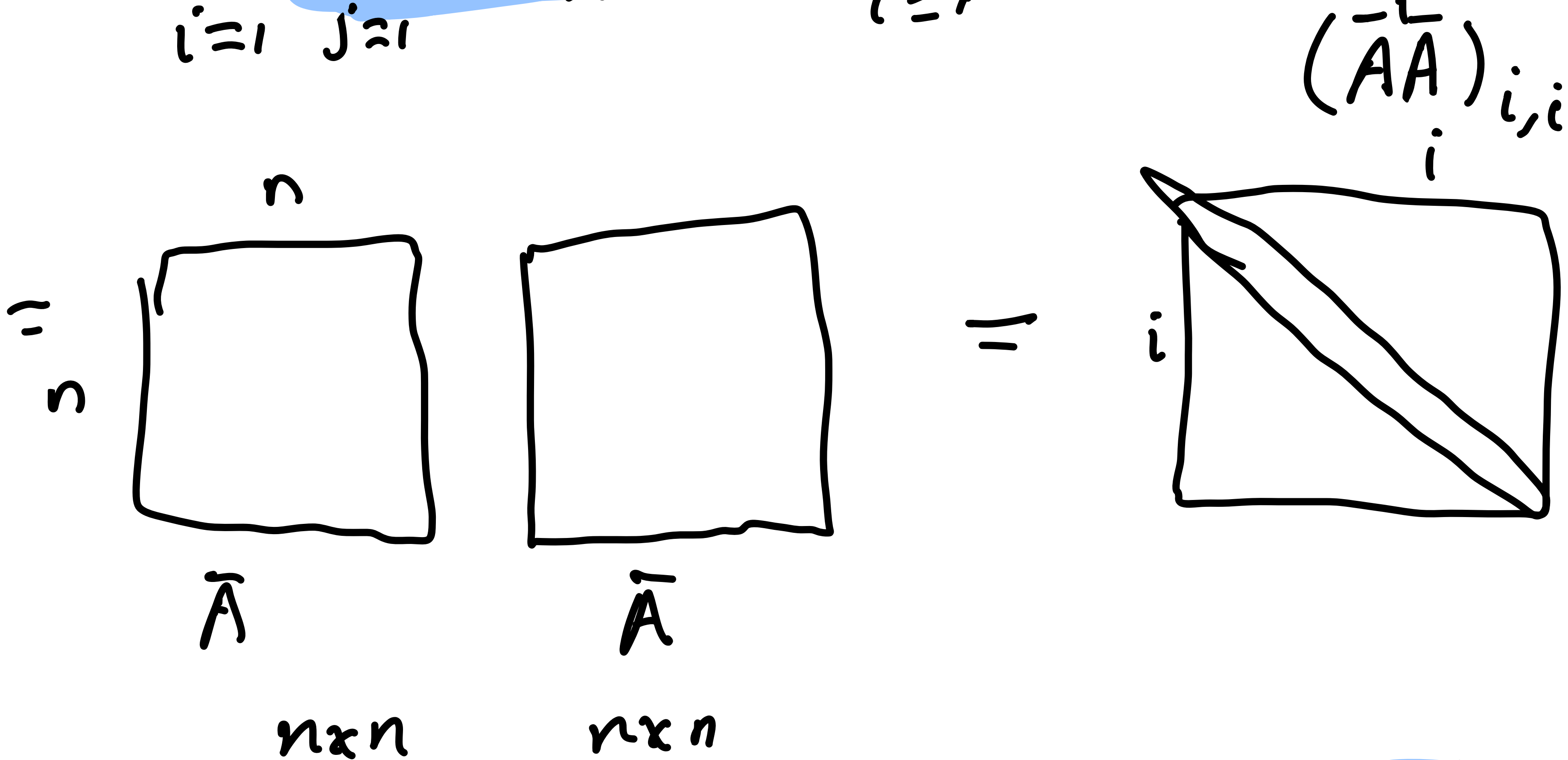
Want to show $\|\bar{A}\|_F^2 = \sum_{i=1}^{n} \lambda_i^2$

Homework

$$\|\bar{A}\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{A}_{i,j}^2 = \sum_{i=1}^{n} (\bar{A}^T \bar{A})_{i,i} = tr(\bar{A}^T \bar{A}) = \sum_{i=}^{n} \lambda_i^2$$

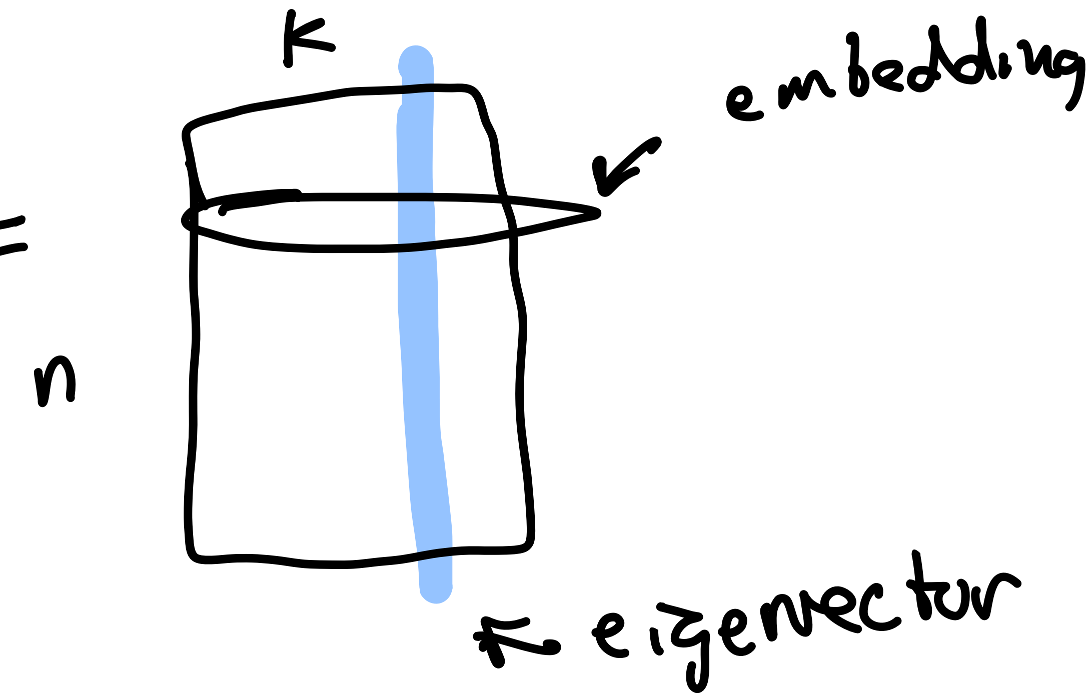
It suffices to show eigenvals of $\bar{A}^T\bar{A}$ are $\lambda_1^2 \le \lambda_2^2 \le \dots \le \lambda_n^2$

$\bar{A}^T \bar{A} =$



$$(\bar{A}^T \bar{A})_{i,i} = \bar{A}_i^T \bar{A}_i = \sum_{j=1}^{n} \bar{A}_{ij} \bar{A}_{ij} = \sum_{j=1}^{n} (\bar{A}_{ij})^2$$

$$F_\Theta =$$



$$F_\Theta F_\Theta^T = \sum_{i=1}^{k} v_i v_i^T \lambda_i$$

$$\arg\min_\Theta \| \bar{A} - F_\Theta F_\Theta^T \|_F^2 =$$

$$\arg\min_\Theta \left\| \sum_{i=1}^{n} v_i v_i^T \lambda_i - \sum_{i=1}^{k} v_i v_i^T \lambda_i \right\|_F^2 =$$

$$\arg\min_\Theta \left\| \sum_{i=k+1}^{n} v_i v_i^T \lambda_i \right\|_F^2 = \sum_{i=k+1}^{n} \lambda_i^2$$

CLIP

$$\mathcal{L}(\theta, \psi) = -2 \mathop{\mathbb{E}}_{x,t \text{ pos}}\left[f_\theta(x)^T g_\psi(t) \right] + \mathop{\mathbb{E}}_{x,t \text{ neg}}\left[\left(f_\theta(x)^T g_\psi(t) \right)^2 \right]$$

A cat and a dog wrapped in blankets

↳ search

↳

x $\quad$ t $\quad$ Positive pair

$f_\theta(x)$ $\quad$ $g_\psi(t)$