Plan

Logistics

Review

Spectral Graph Theory

Michael's talk

Warner 101 at noon today

Project

Friday 2nd

1. Codebase (12 points)
2. Report (6 points)
3. Presentation
   - ↳ each person records whole talk 2x

Problems

↳ work together

↳ ask questions

$$X = U \Sigma V^T \qquad X \in \mathbb{R}^{n \times d}$$

$$X^T X = V \Sigma^T U^T U \Sigma V^T$$

$$= V \Sigma \Sigma V^T$$

$$\triangleq M$$

$$X V = U \Sigma V^T V = U \Sigma$$

Goal: Find top eigenvector fast

Power Method

$z^{(0)} \sim N(0, I)$

$z^{(0)} = z^{(0)} / \|z\|_2$

for $t = 1, \dots, q$

$\quad z^{(t)} = M z^{(t-1)}$

$\quad n_t = \|z^{(t)}\|_2$

$\quad z^{(t)} = z^{(t)} / n_t$

return $z^{(q)}$

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$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

$$\underset{d \times d}{M} = \sum_{i=1}^{d} \underset{d\times 1}{v_i} \underset{1 \times d}{v_i^T} \sigma_i^2$$

$$M v_k = \sum_{i=1}^{d} v_i v_i^T v_k \sigma_i^2 = v_k \sigma_k^2$$

$$z^{(t)} = \sum_{j=1}^{d} v_j c_j^{(t)}$$

$$z^{(t+1)} = \frac{M z^{(t)}}{n_t} = \sum_{i=1}^{d} v_i v_i^T \sigma_i^2 \sum_{j=1}^{d} v_j c_j^{(t)} \cdot \frac{1}{n_t}$$

$$= \sum_i \sum_j v_i v_i^T v_j c_j^{(t)} \sigma_i^2 \cdot \frac{1}{n_t}$$

$$= \sum_i v_i c_i^{(t)} \sigma_i^2 \cdot \frac{1}{n_t}$$

$$= \left(v_1 \frac{c_1^{(t)}}{n_t} \sigma_1^2 + v_2 \frac{c_2^{(t)} \sigma_2^2}{n_t} + \dots + \right)$$

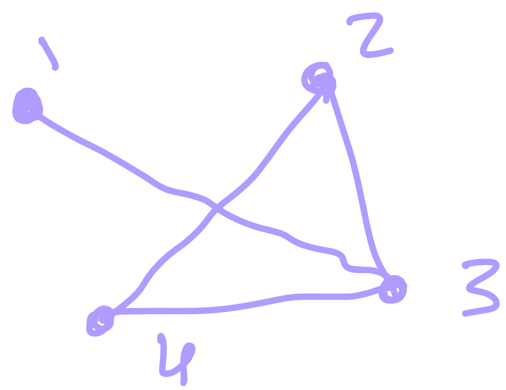
$$c_i^{(q)} = c_i^{(0)} \cdot \frac{(\sigma_i^2)^q}{\prod_{\ell=1}^{q} n_\ell}$$

Spectral Graph Theory

n nodes

$G = (V, E)$

m edges



Adjacency A

$$A_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$$

Degree D

$$D_{i,j} = \begin{cases} d_i & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$D = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 2 \end{bmatrix} \end{array}$$

$$L = D - A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \end{array}$$

Laplacian

Normalized

$$\bar{A} = D^{-1/2} A D^{-1/2}$$

$$\bar{L} = D^{-1/2} D D^{-1/2} - D^{-1/2} A D^{-1/2}$$

$$= I - \bar{A}$$

$B \in \mathbb{R}^{m \times n}$ edge-incidence

$$B_{(i,j),k} = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = j \\ 0 & \text{else} \end{cases}$$

$$b_{(i,j)} = \begin{array}{c} \\ \left[\right. \end{array} \begin{array}{ccccccc} & & i & & & j & \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{array} \begin{array}{c} \\ \left. \right] \end{array}$$

$$B = \begin{bmatrix} — & b_{(i,j)} & — \\ — & b_{(i',j')} & — \\ & \vdots & \end{bmatrix}$$

$m \times n$

$$L = B^T B = \begin{array}{c} \\ j \end{array} \begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix} \begin{array}{c} i \\ \begin{bmatrix} — & — & — \\ — & — & — \\ — & — & — \end{bmatrix} \end{array}$$

$n \times m$ $\quad$ $m \times n$

(1) example $(n \leq 4)$

(2) generality

$$x^T L x$$

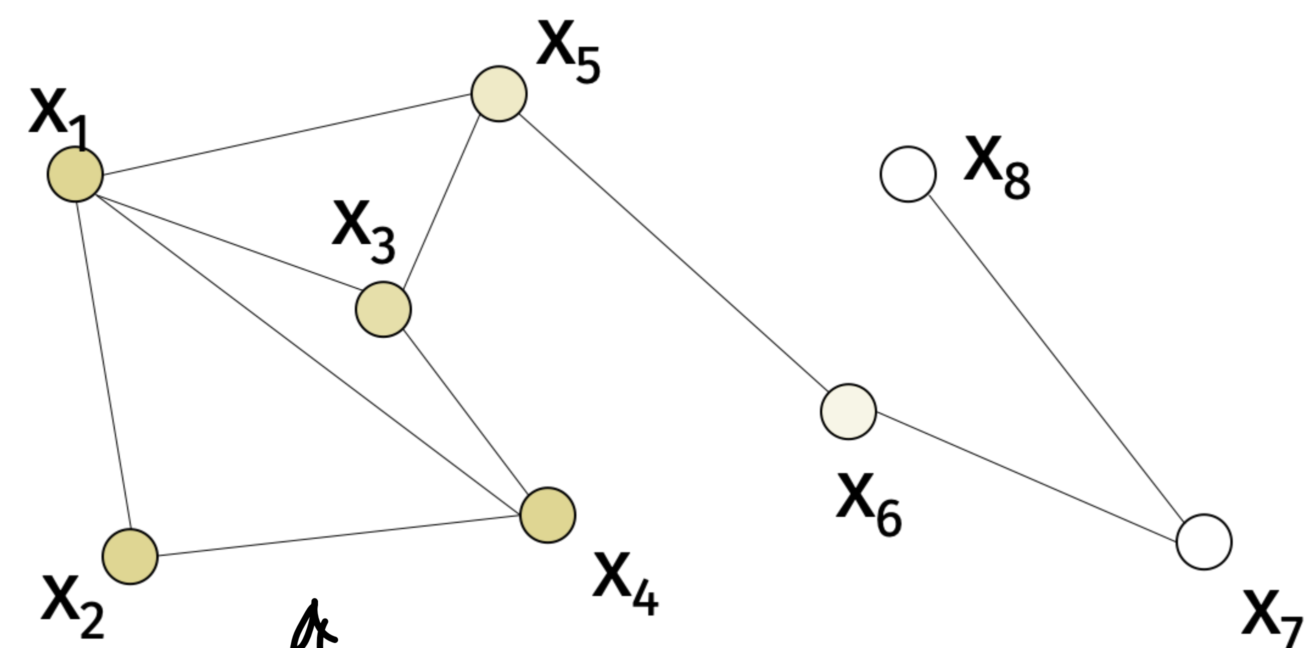
$$= x^T B^T B x$$

$$= \|Bx\|_2^2$$

$$= \left\| \underset{B}{\begin{array}{cc} \begin{array}{cccccc} \; & \; & \; & i & j & \; \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ & & & & & \end{bmatrix} & (i,j) \end{array}} \; \underset{x}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} \right\|_2^2$$

$$= \left\| \underset{m}{\begin{array}{cc} \begin{bmatrix} x_i - x_j \\ \vdots \\ \\ \end{bmatrix} & \begin{array}{c} (i,j) \\ \\ \\ \end{array} \end{array}} \right\|_2^2$$

$$f(x) = x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2$$



small changes between adjacent nodes

$v_1, \dots, v_n$ eigenvectors L

Courant - Fischer min-max principle

$$Lv = \lambda v$$

$$v_n = \underset{v: \|v\|_2 = 1}{\operatorname{argmin}} \quad v^T L v \qquad\qquad v^T L v = v^T v \; \lambda = \lambda$$

$$v_n = \mathbb{1}$$

$$v_{n-1} = \underset{v: \|v\|_2 = 1, \; \langle v_n, v \rangle = 0}{\operatorname{argmin}} \quad v^T L v$$

$$\Updownarrow$$

$$\langle \mathbb{1}, v \rangle = 0$$

Spectral Clustering

↳ Social network

↳ Machine learning

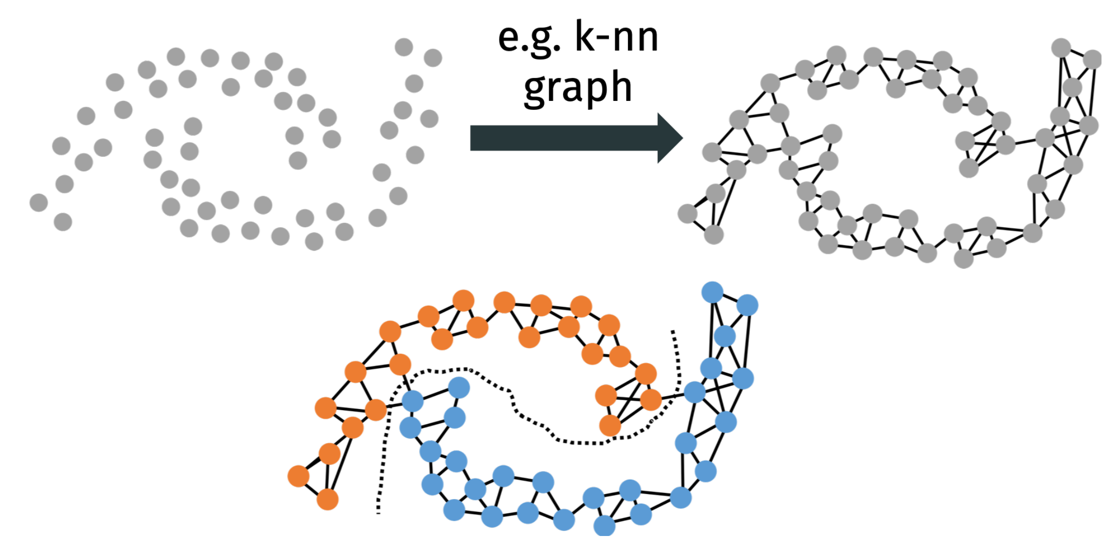
↳ Graph visualization

Goal: Partition into $S \subseteq V$,
$S^c = V \setminus S$

$c \in \{-1,1\}^n$ cut indicator

$$
\begin{aligned}
c^T L c &= c^T B^T B c \\
&= \sum_{(i,j)\in E} (c_i - c_j)^2 \\
&= 4 \ \mathrm{cut}(S, S^c)
\end{aligned}
$$

\# edges separated



Balanced Cut

$$\min \quad c^T L c \quad \text{s.t.} \quad c^T \mathbf{1} = 0$$

$$c \in \{-1/\sqrt{n}, 1/\sqrt{n}\}^n$$

$$c^T L c = 4 \cdot \mathrm{cut}(S, S^c) \cdot \frac{1}{n}$$

$$|c^T \mathbf{1}| = \big|\, |S| - |S^c| \,\big| \cdot \frac{1}{\sqrt{n}}$$

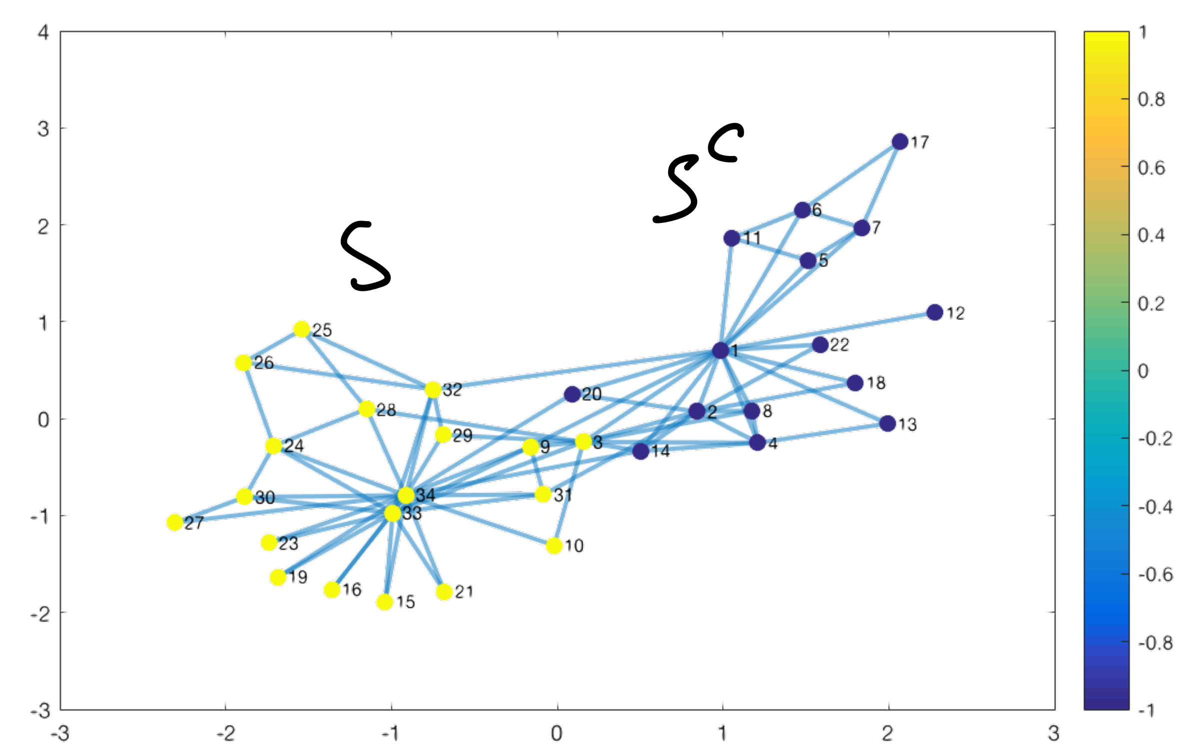
Relaxed Balanced Cut

$$\min \quad c^T L c \quad \text{s.t.} \quad c^T \mathbf{1} = 0$$

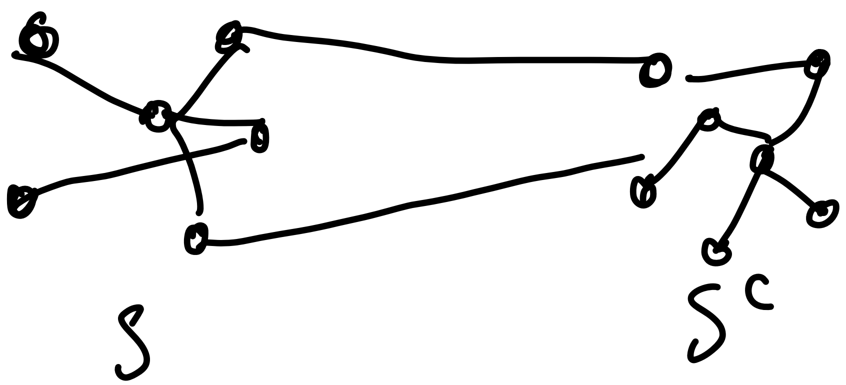
$$\|c\|_2 = 1$$

$S = \{ i : v_{n-1}[i] \geq 0 \}$

v_{n-1} = 2^{nd} smallest eigenvector of L



Stochastic Block Model



$Pr(\text{edge within}) = p = .5$

$Pr(\text{edge between}) = q = .1$

$$\mathbb{E}[A] = \begin{array}{c c} & \begin{array}{cc} \overbrace{\quad\quad\quad\quad}^{S} & \overbrace{\quad\quad\quad}^{S^c} \end{array} \\ \begin{array}{c} S\{ \\ \\ S^c\{ \\ \\ \end{array} & \begin{bmatrix} p & p & p & p & q & q & q \\ p & p & p & p & q & q & q \\ p & p & p & p & q & q & q \\ q & q & q & q & p & p & p & p \\ q & q & q & q & p & p & p & p \\ & & & & p & p & p & p \end{bmatrix} \end{array}$$

$n \times n$

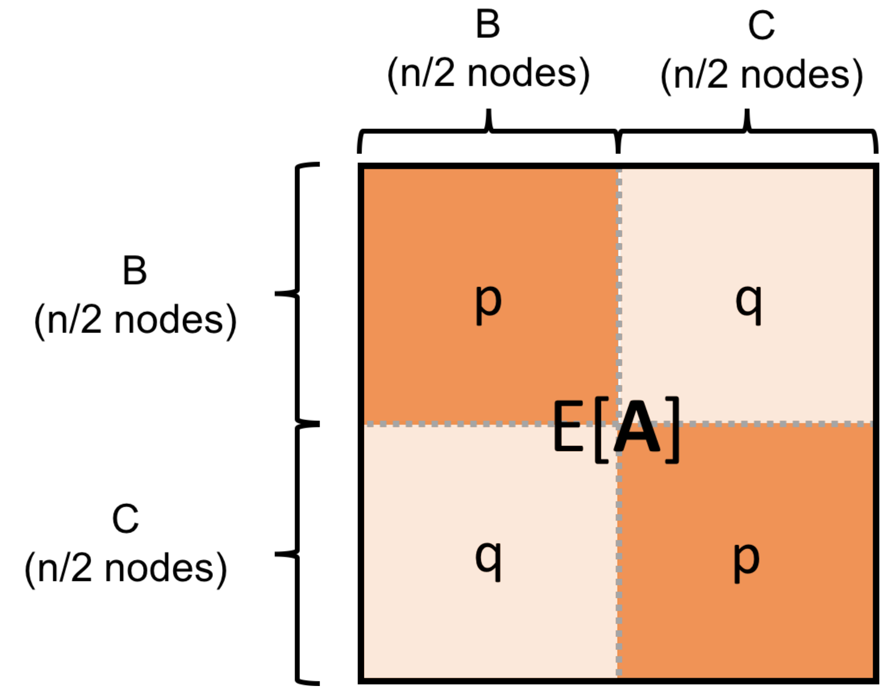
$$\mathbb{E}[A] = V \Lambda V^T$$

$$\mathbb{E}[A] = v_1 v_1^T \lambda_1 + v_2 v_2^T \lambda_2$$

$$\lambda_1 v_1 v_1^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \lambda_1$$

$$v_1 v_1^T = \begin{bmatrix} \lambda_1 & \lambda_1 & \lambda_1 & \cdots \\ \vdots & & & \\ \vdots & & & \end{bmatrix}$$

$$E[A] = V \Lambda V^T$$

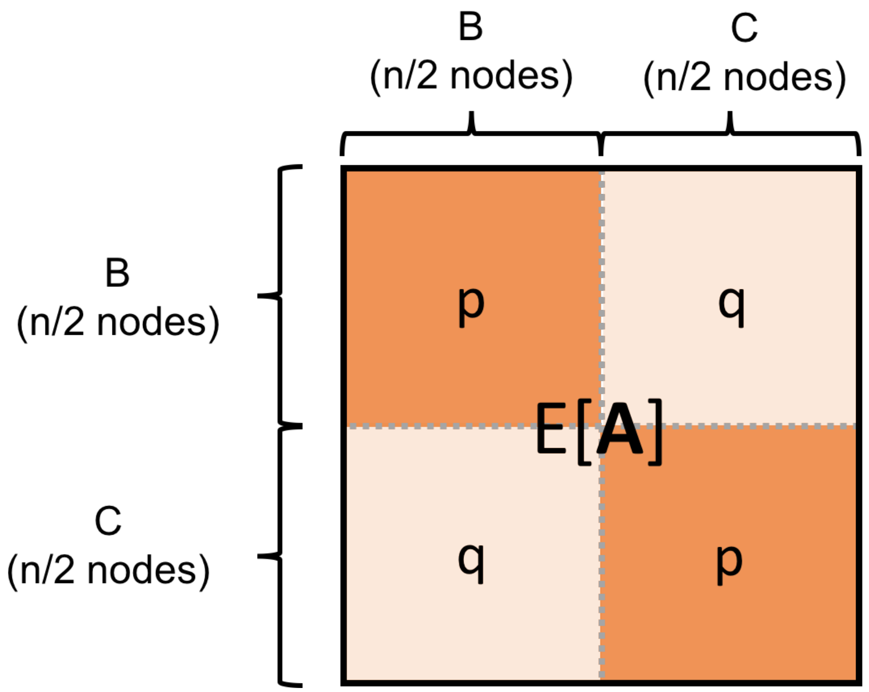


$$- \begin{bmatrix} \lambda_1 & \lambda_1 & \\ \vdots & \lambda_1 & \ddots \\ \vdots & & \end{bmatrix}$$

$$E[A] = v_1 v_1^T \lambda_1 + v_2 v_2^T \lambda_2$$

$$v_2 v_2^T = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

$$\underbrace{E[A] - v_1 v_1^T \lambda_1}_{\begin{bmatrix} p-\lambda_1 & q-\lambda_1 \\ q-\lambda_1 & p-\lambda_1 \end{bmatrix}} = v_2 v_2^T \lambda_2 = \left[\begin{array}{cccccc|cccc} 1 & 1 & 1 & 1 & 1 & & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & & -1 & -1 & -1 & \\ 1 & 1 & 1 & 1 & & & & & & \\ \hline -1 & -1 & -1 & -1 & & & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & & & 1 & 1 & 1 & 1 \end{array}\right] \lambda_2$$



$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & \end{bmatrix} \lambda_1 + \left[\begin{array}{c|c} 1 \; 1 \; \cdots & -1 \; -1 \\ 1 \; 1 \; \cdots & -1 \; -1 \\ & -1 \\ \hline -1 \; -1 & 1 \; 1 \; 1 \\ -1 & 1 \; 1 \; \cdots \end{array}\right] \lambda_2$$

$$1 \cdot \lambda_1 + 1 \cdot \lambda_2 = p \Rightarrow (\lambda_2 + q) + \lambda_2 = p \Rightarrow \lambda_2 = \frac{p-q}{2}$$

$$1 \cdot \lambda_1 + -1 \cdot \lambda_2 = q \Rightarrow \lambda_1 = \lambda_2 + q \Rightarrow \lambda_1 = \frac{p-q}{2} + \frac{2q}{2} = \frac{p+q}{2}$$