Plan

Logistics
Review
Course Response Forms
Sparse Recovery

Games @ 6pm Wednesday
↳ Pizza
↳ Extra Credit
(name and # points)

PSet 3

Nice Job!
Chat!! :)

PSet 4

I'm sorry
Today: evaluation

Project

↳ work hard tomorrow
↳ play hard tomorrow night
↳ presentation (recorded links)
↳ Gradescope

Review

$A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$

$$x^* = \underset{x}{\operatorname{argmin}} \|Ax - b\|_2^2$$
$$= (A^T A)^{-1} A^T b$$

$O(nd^2)$ time

$\Pi \in \mathbb{R}^{m \times n}$ $\quad m \approx d$

$$\tilde{x} = \underset{x}{\operatorname{argmin}} \|\Pi A x - \Pi b\|_2^2$$
$$= (A^T \Pi^T \Pi A)^{-1} A^T \Pi^T \Pi b$$

$O(md^2)$

$$\|A\tilde{x} - b\|_2^2 \le (1+\epsilon) \|Ax^* - b\|_2^2$$

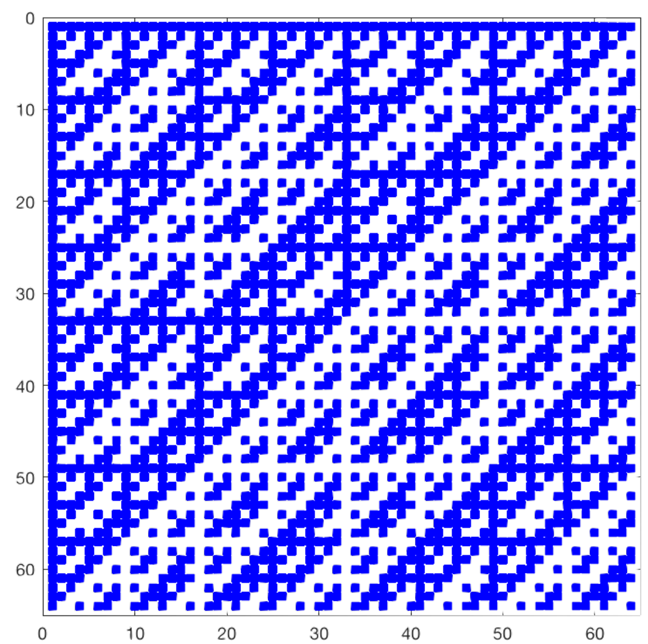
Compute ΠA, Πb fast

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_d \\ | & | & & | \end{bmatrix}$$

$$\underset{m \times n}{\Pi} a_i = S \underset{m \times n}{H} D a_i$$

$$\begin{bmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix} \underset{a}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \begin{bmatrix} +1 \\ -1 \\ +1 \\ +1 \end{bmatrix} \underset{\text{elementwise}}{*} \underset{a}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}$$

H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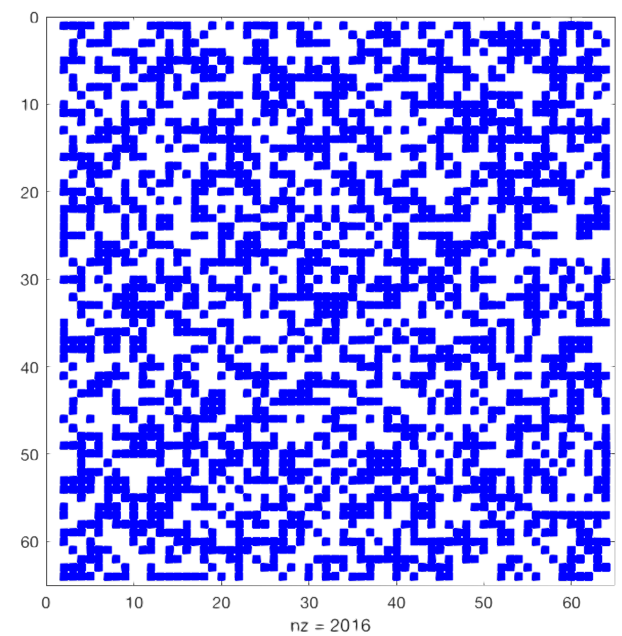
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nz = 2016

Course Response Forms

What did you like? What can be improved?

- Content and difficulty
- Group activities
- Accessibility /receptiveness
- LaTeX, self-grade

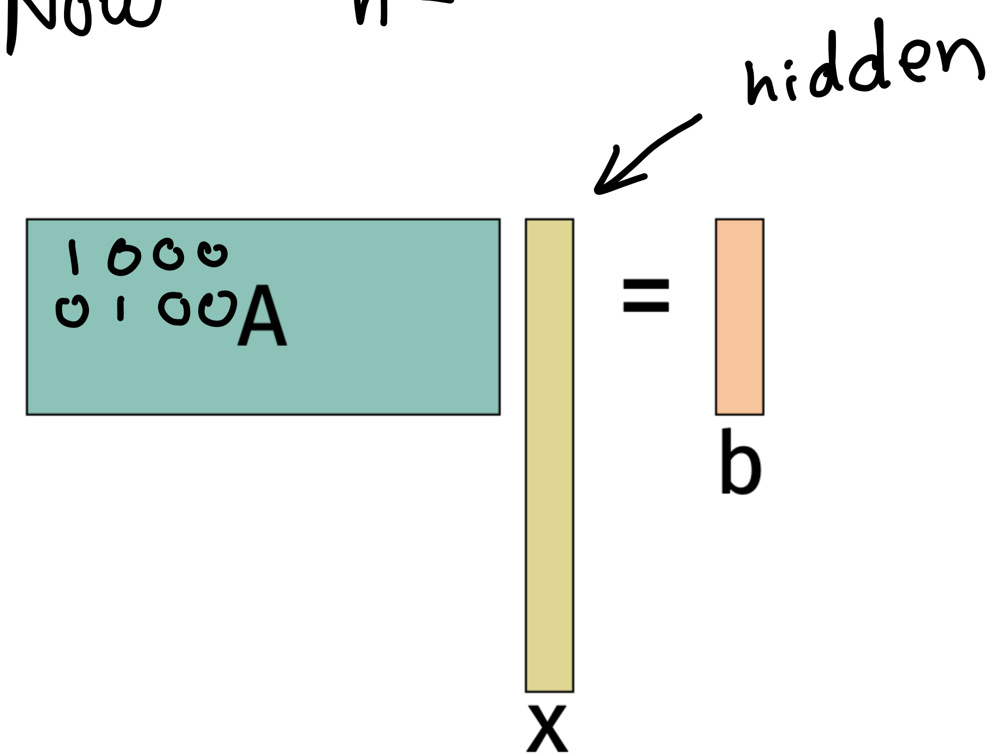
- Daily check in forms
- Review the next day
- Afternoon problem solving
- Typed notes, slides

Sparse Recovery

$A \in \mathbb{R}^{n \times d}$ $\quad x \in \mathbb{R}^d$ $\quad b \in \mathbb{R}^n$

$Ax = b$

Now $\quad n << d$



Goal: Recover x by choosing A and observing $Ax = b$

Trivial solution

$n = d$ then I can recover x exactly

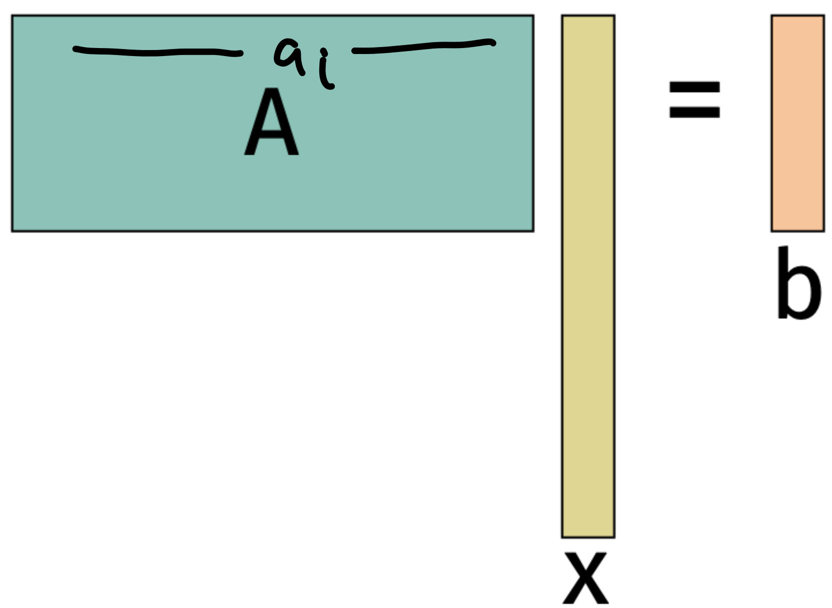
$A \in \mathbb{R}^{d \times d}$ is too large

Assume x is k-sparse

$\|x\|_0 = \#$ non-zeros in $x \le k$

Goal: Recover k-sparse x
with only a few "measurements"

$n = O(k \log k)$ measurements



$\langle a_i, x \rangle = b_i$

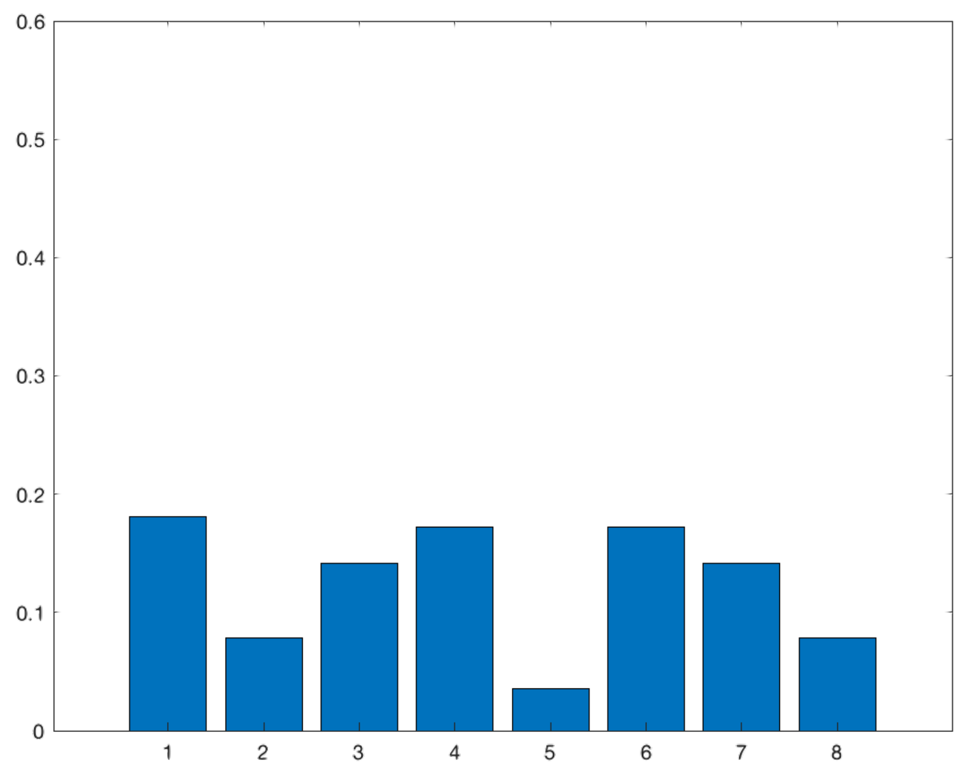
measurement

Applications

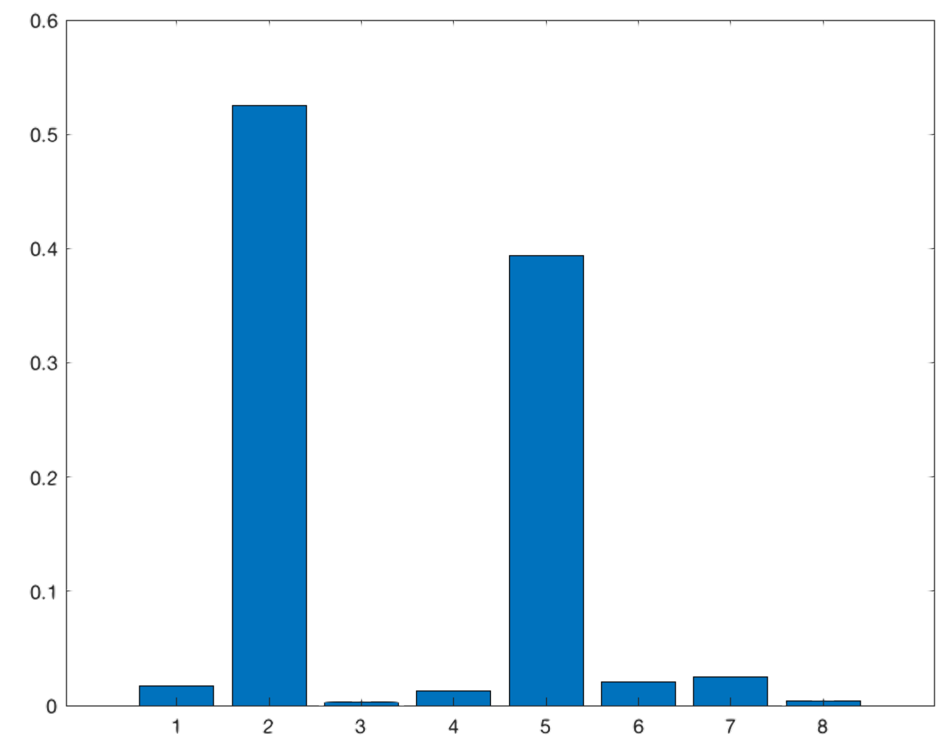
- Compress images because Fourier transform is sparse
- Parameters that fit "Occam's razor"
- X-rays, MRI
- Earth exploration

Fourier Transform

Data



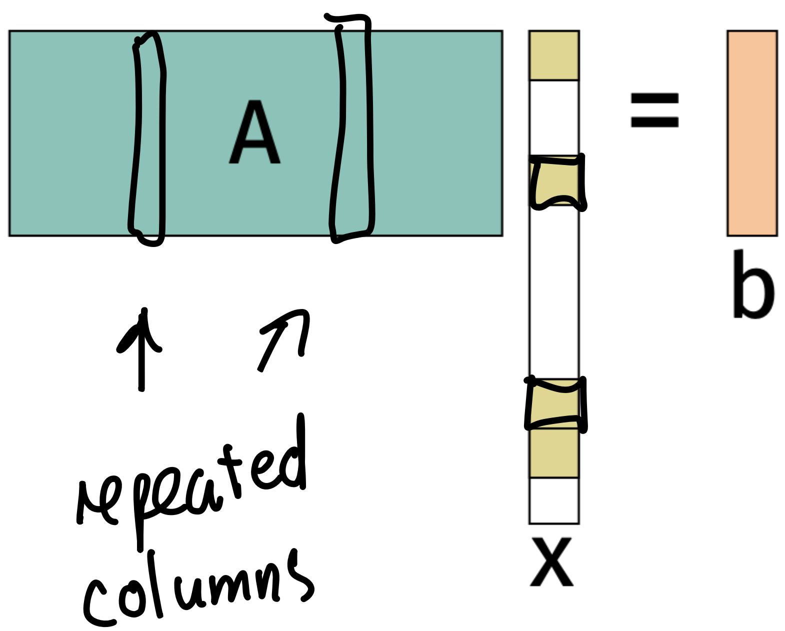
Frequency



Uncertainty Principle

Which A work?

Which definitely do not?



(k,ϵ)- Restricted Isometry Property

A satisfies (k,ϵ)-RIP if for all x with $\|x\|_0 \le k$,

$$(1-\epsilon)\|x\|_2^2 \le \|Ax\|_2^2 \le (1+\epsilon)\|x\|_2^2$$

↪ Looks like JL Lemma

↪ Preserves norm of k-sparse (rather than discrete set or subspace)

$A \in \mathbb{R}^{n \times d}$

$$n = O\left(\frac{k \log d + \log(1/\delta)}{\epsilon^2}\right)$$

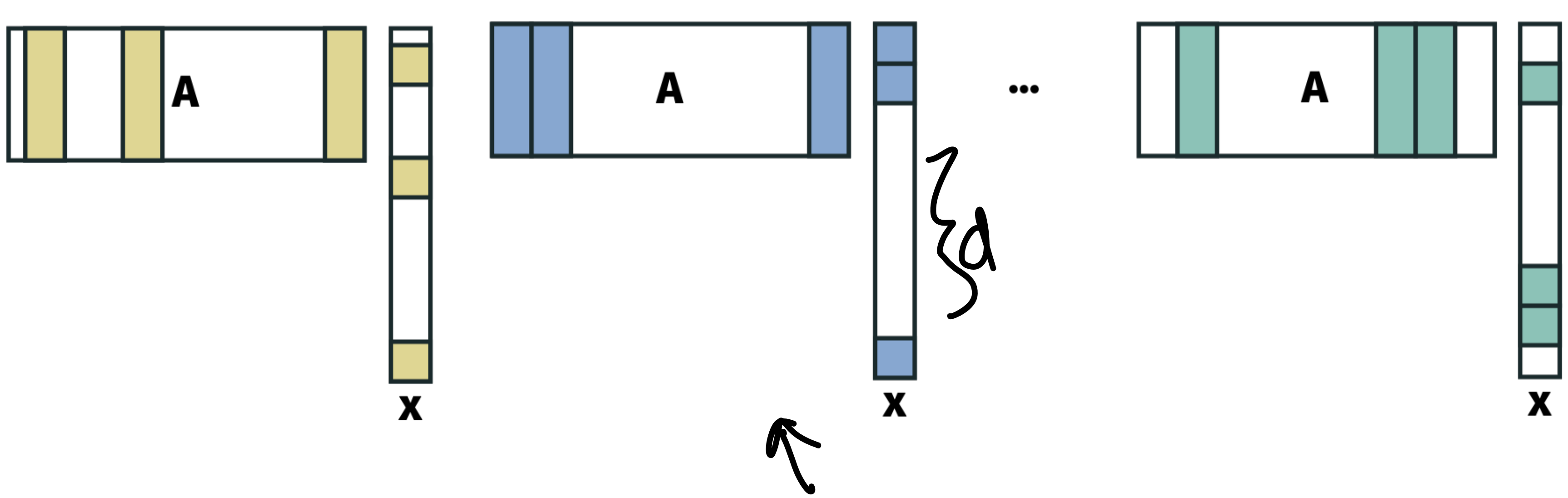
$$T = \binom{d}{k}$$

$$\delta' = \frac{\delta}{T}$$

Consider $S_k = \{ x : \|x\|_0 \le k \}$

$\log(1/\delta')$

$S_k = U_1 \cup U_2 \cup U_3 \cup \dots \cup U_T$



Theorem: If A is $(2k, \epsilon)$-RIP for $\epsilon < 1$

then x is the unique minimizer

$$\min_z \|z\|_0 \text{ s.t. } Az = b$$

$\leftarrow O(d^k)$ time to find

$\|y-x\|_2^2 = 0$

$= \sum_{i=1}^{d} (y_i - x_i)^2$

$\Rightarrow x = y$

Proof: Suppose for contradiction that there is a different y with

$\|y\|_0 \le \|x\|_0 \le k$ and $Ay = b$

$y \neq x \quad Ay = b \quad Ax = b$

$Ay - Ax = A(y-x)$
$= b - b = 0$

$\|w\|_2^2(1-\epsilon) \le \|Aw\|_2^2$

$\|y-x\|_2^2(1-\epsilon) \le \|A(y-x)\|_2^2$
$= 0$

Theorem: If A is $(3k, \epsilon)$-RIP for $\epsilon < 1$

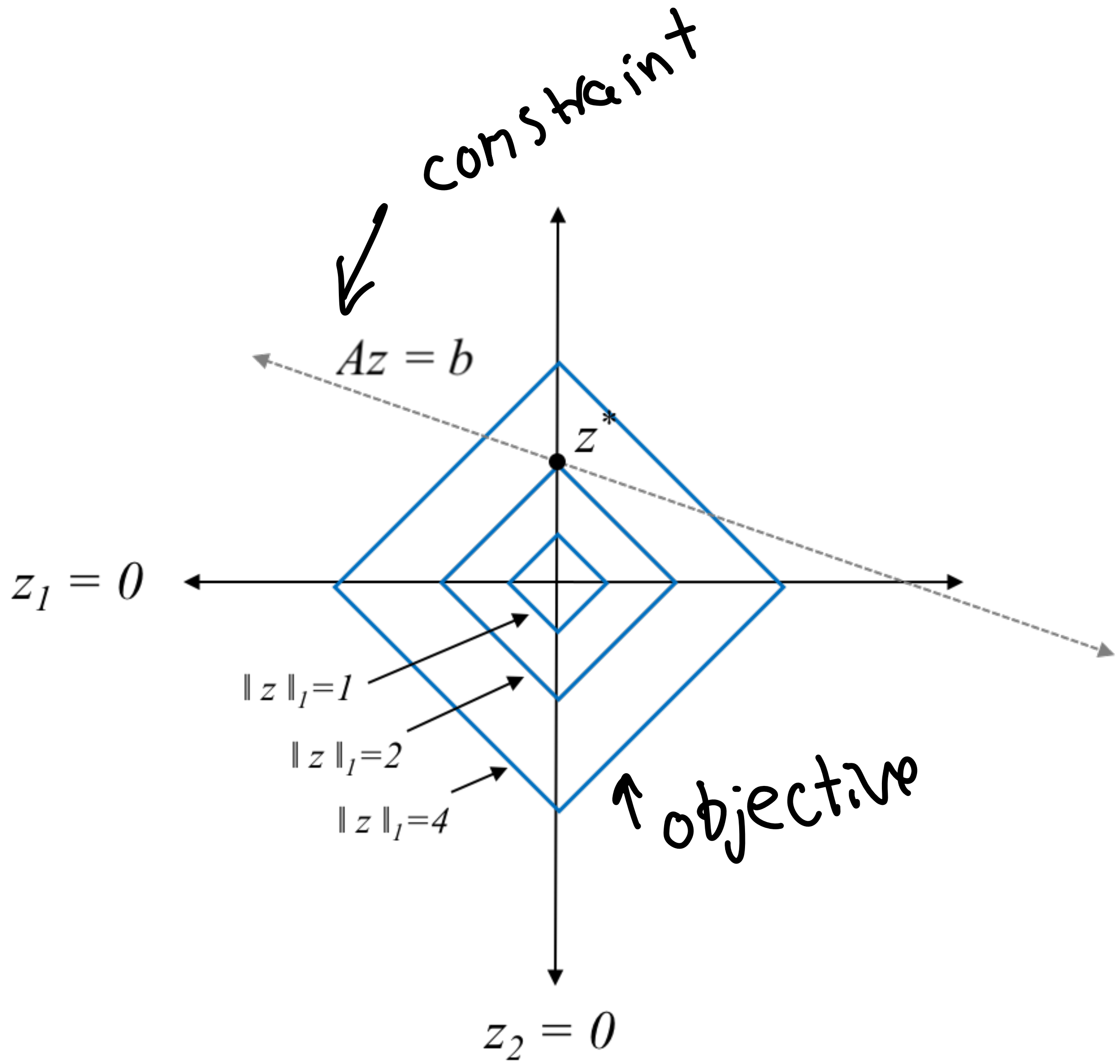
then x is the unique minimizer

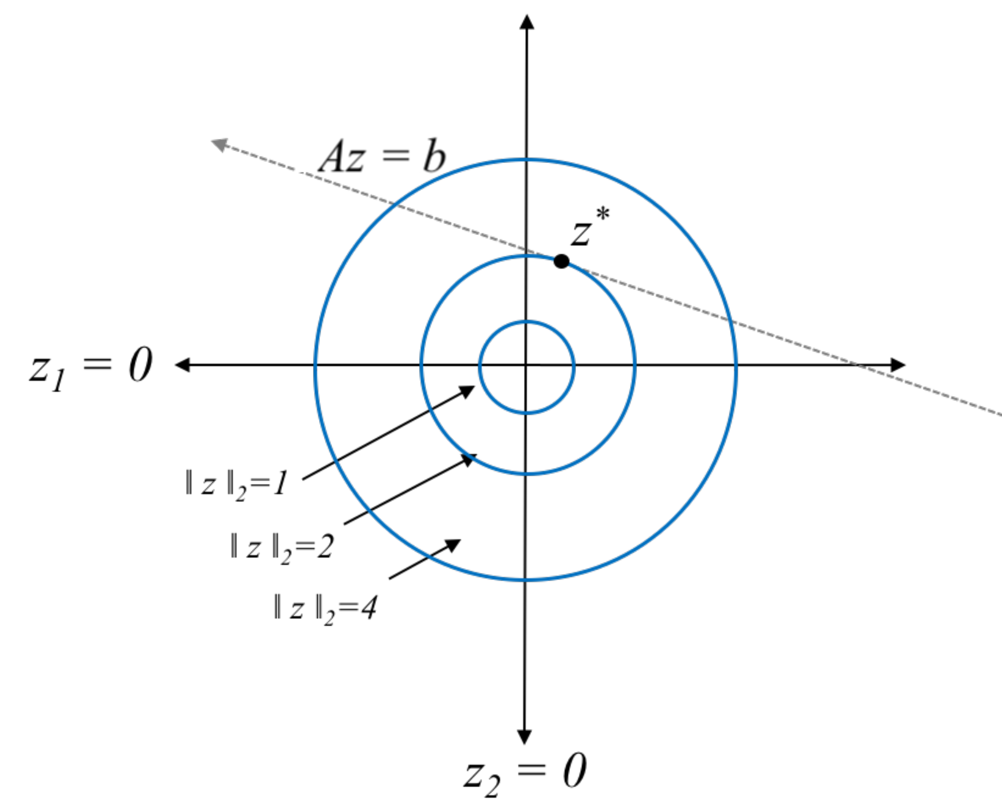
$$\min_{z} \|z\|_1 \text{ s.t. } Az = b$$

convex optimization solve with linear programming $O(d^{3.5})$

☆ Exponentially faster

☆ Like relaxation


constraint
$Az = b$
z^*
$z_1 = 0$
$\| z \|_1 = 1$
$\| z \|_1 = 2$
$\| z \|_1 = 4$
objective
$z_2 = 0$


$Az = b$
z^*
$z_1 = 0$
$\| z \|_2 = 1$
$\| z \|_2 = 2$
$\| z \|_2 = 4$
$z_2 = 0$