Tuesday, April 17

Lanczos!

Spectral graph theory!

Krylov Supspace

Power method just uses last vector. Can we do better?

$$K = \{ z^{(0)}, Az^{(0)}, A^2 z^{(0)}, \dots, A^{m-1} z^{(0)} \}$$

Lanczos Method

$z^{(0)} \sim \mathcal{N}(0, I)$ initial start

Compute $Q \in \mathbb{R}^{d \times m}$, an orthonormal basis for K

Form $T = Q^T A Q$ (T is tridiagonal, prove for hw)

Let z be top eigenvector of T

Return $y^* = Qz$

Analysis:

1. $y^* = Qz$ is the best vector in $\mathcal{K}$ i.e.,

$$y^* = \underset{y}{\text{argmin}}\ \|X - Xyy^T\|_F^2$$
$$= \underset{y}{\text{argmin}}\ \|X\|_F^2 - \|Xyy^T\|_F^2 =$$

2. There is a very "good" vector w in $\mathcal{K}$. All w in $\mathcal{K}$

$$w = \alpha_1 z^{(0)} + \alpha_2 A z^{(0)} + \dots + \alpha_m A^{m-1} z^{(0)}$$
$$= p(A)\ z^{(0)}$$
$$= \sum_{i=1}^{d} p(\lambda_i)\ v_i v_i^T\ z^{(0)}$$
$$= \sum_{i=1}^{d} p(\lambda_i)\ v_i\ c_i^{(0)}$$

Goal: Find $p(\lambda)$ s.t. $p(\lambda_j)$ small for $j \neq i$ but $p(\lambda_1)$ large

Answer: Chebyshev polynomial!

Claim: There is degree $m = O(\sqrt{1/\gamma}\log\frac{d}{\epsilon})$ Chebyshev p where $p(1) = 1$ but $|p(\lambda)| \leq \epsilon$ for $0 \leq \lambda \leq 1-\gamma$

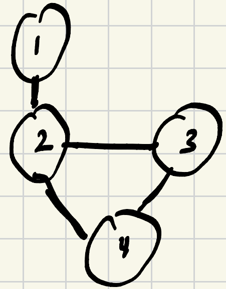
Spectral Graph Theory

Graph data:

- ↳ social networks
- ↳ internet

$$\underbrace{G}_{\text{graph}} = (\underbrace{V}_{\substack{\text{vertices}\\(n)}}, \underbrace{E}_{\substack{\text{edges}\\(m)}})$$

assume undirected for now



$V = \{1, 2, 3, 4\}$

$E = \{(1,2), (2,3), (2,4), (3,4)\}$

Adjacency Matrix

$A \in \mathbb{R}^{n \times n}$

$$[A]_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ or } (j,i) \in E \\ 0 & \text{else} \end{cases}$$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Degree Matrix

$D \in \mathbb{R}^{n \times n}$

$$[D]_{i,i} = \sum_{j=1}^{n} [A]_{i,j}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Laplacian

$L = D - A$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Fun Facts:

- If L has k eigenvalues equal to 0, then G has k connected components
- Sum of cubes of eigenvalues equals number of triangles times 6

Edge-Incidence Matrix

$B \in \mathbb{R}^{m \times n}$

$$B_{(i,j),k} = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = j \\ 0 & \text{else} \end{cases}$$

Show:

- $L = B^T B$
- $= \sum_{i=1}^{m} b_i b_i^T$ where b_i is i^{th} row of B

Thursday, April 9

Graph $G = (V, E)$ with n nodes/vertices, m edges

undirected

Adjacency Matrix: $A \in \mathbb{R}^{n \times n}$ $\quad [A]_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ or } (j,i) \in E \\ 0 & \text{else} \end{cases}$

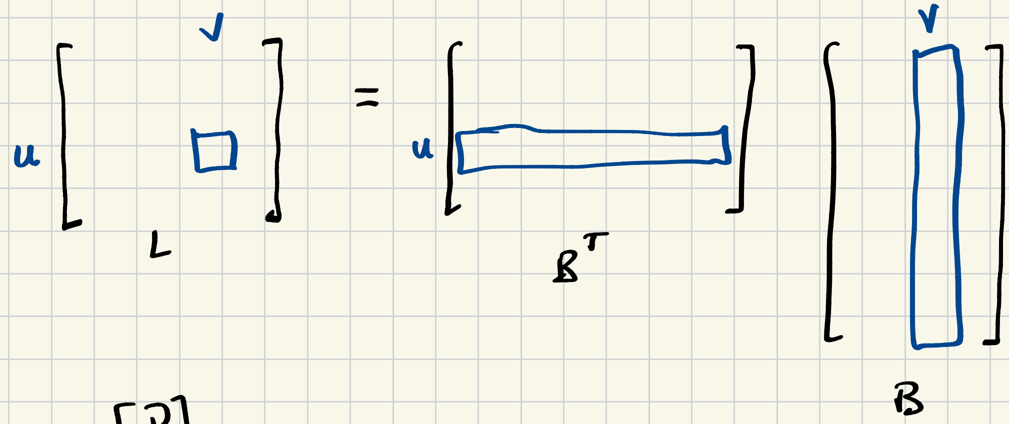
Degree Matrix: $D \in \mathbb{R}^{n \times n}$ $\quad [D]_{i,i} = \sum_{j=1}^{n} [A]_{i,j} = \#\text{edges to } i$

Laplacian Matrix: $L \in \mathbb{R}^{n \times n}$ $\quad L = D - A$

Edge-incidence Matrix: $B \in \mathbb{R}^{m \times n}$ $\quad [B]_{(i,j),u} = \begin{cases} 1 & \text{if } i = u \\ -1 & \text{if } j = u \\ 0 & \text{else} \end{cases}$

$$= \mathbb{1}[i = u] - \mathbb{1}[j = u]$$

$$\boxed{\text{Claim: } L = B^T B}$$



$$[B^T B]_{u,v} = \sum_{(i,j)\in E} [B^T]_{u,(i,j)} \, [B]_{(i,j),v}$$

$$= \sum_{(i,j)\in E} \left(\mathbb{1}[i=u] - \mathbb{1}[j=u]\right)\left(\mathbb{1}[i=v] - \mathbb{1}[j=v]\right)$$

if $u=v$

$$= \sum_{(i,j)\in E} \mathbb{1}[i=u] + \mathbb{1}[j=u] = \#\text{ edges to } u = [D]_{u,u} = [L]_{u,v}$$

if $u \neq v$

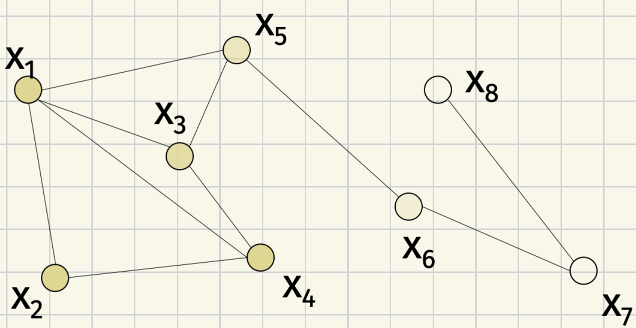
$$= -\mathbb{1}[i=u]\,\mathbb{1}[j=v] - \mathbb{1}[j=u]\,\mathbb{1}[i=v] = -[A]_{u,v} = [L]_{u,v}$$

Insights

$$x^T L x = x^T B^T B x = \left\| \begin{array}{c} \\ (i,j) \\ \\ \end{array} \overset{B}{\begin{bmatrix} & \\ +1 & -1 \\ & \end{bmatrix}} \overset{x}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} \right\|_2^2 = \sum_{(i,j)\in E} (x_i - x_j)^2 \geq 0$$

If $B = U \Sigma V^T$ then $L = V \Sigma^2 V^T$.

Consider $f(x) = x^T L x = \begin{cases} \text{small if } x \text{ "smooth"} \\ \text{large if } x \text{ "jumpy"} \end{cases}$ on graph



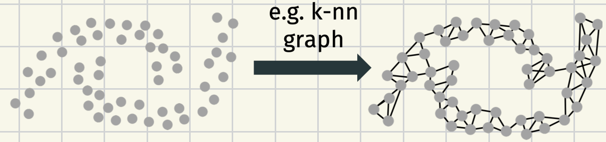
Graph Partitioning

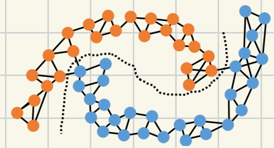
↳ understanding social networks

↳ clustering

↳ visualization

↳ Karate graph!! :)





Let $c \in \{-1, 1\}^n$ be "cut" vector.

i.e. $c_i = \begin{cases} 1 & \text{if } i \in S \\ -1 & \text{else} \end{cases}$

Goal: Find S and $S^c = V \setminus S$ s.t.

1. Number of edges btwn S, S^c small

2. S, S^c are not too small

1. $c^T L c = \sum_{(i,j) \in E} (c_i - c_j)^2$

$= 4 \cdot \#$ edges between S, S^c

$= 4 \, \text{cut}(S, S^c)$

2. $|c^T \mathbb{1}| = \big| |S| - |S^c| \big|$

Computing the Optimal Cut

$$\min_{c \in \{-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\}^n} c^T L c \quad \text{s.t.} \quad c^T \mathbb{1} = 0$$

$\Downarrow$ relax

$$\min_{c : \|c\|_2 = 1} c^T L c \quad \text{s.t.} \quad c^T \mathbb{1} = 0$$

Eigen Perspective

$$\lambda_d = \min_{v : \|v\|_2 = 1} v^T L v \qquad v_d = \operatorname*{argmin}_{v : \|v\|_2 = 1} v^T L v$$

$$\lambda_{d-1} = \min_{\substack{v : \|v\|_2 = 1 \\ \langle v, v_d \rangle = 0}} v^T L v \qquad v_{d-1} = \operatorname*{argmin}_{\substack{v : \|v\|_2 = 1 \\ \langle v, v_d \rangle = 0}} v^T L v$$

Show:

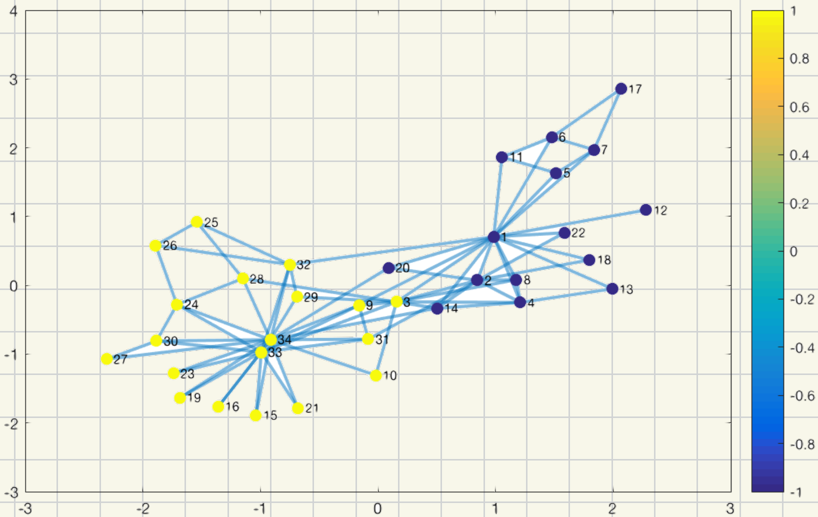
- ↳ Smallest eigenvector of L is $\frac{1}{\sqrt{n}}\mathbb{1}$
- ↳ Second smallest is solution to the relaxed problem

Strategy:

Compute v_{n-1}

Use $c = \text{sign}(v_{n-1})$

Time Complexity:



Stochastic Block Model

Recall $L = D - A$

i.e. top eigenvalue of A

is bottom eigenvalue of L

Motivation: Interesting setting we can analyze

$0 < q < p < 1$

Group S and S^c partition V into two equal groups

$$\Pr(\text{edge } i,j) = \begin{cases} p & \text{if } i,j \in S \text{ or } i,j\ S^c \\ q & \text{else} \end{cases}$$

$$\mathbb{E}[A] = \begin{array}{c c} & \overbrace{\qquad}^{S}\overbrace{\qquad}^{S^c} \\ \begin{array}{c} S \\ S^c \end{array} & \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix} \end{array} = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T = \dots$$